Lecture No -27
Laplace Transforms

$L\{e^{3t}\} = \frac{1}{\ }$; $L\ \{t^3\} = \frac{3!}{\ }$; $L\{2\sin 3t + \cos 3t\} = \frac{s+6}{\ }$

And

s − 3

$L^{-1}\{\frac{1}{\ }\} = {}^{3t}$

s^4

$_{-1}\ 3!$

3 -1

$\frac{s+6}{s^2+9}$

e ; L { } $= t$ $s - 3$

; L { $_{s^2+9}$ } $= 2\sin 3t + \cos 3t$

s^4

We show that

$L\{t^3\} = \frac{3!}{s^4}$

For this consider the integral

$1 \quad \frac{e^{-st}}{-s} \quad \infty \quad 3\ \infty$

$L(t^3) = t^3 e^{-st} dt = [t^3 \ ]^{\infty} - 3t^2 e^{-st} dt = + t^2 e^{-st} dt \int$ $\quad -s \quad 0 \quad \int$

$-s \quad se^{st} \quad 0 \quad s$

0 0 0

= 0 +

$\frac{3}{\ }\{t^2 \frac{e^{-st}}{\ }$ ∞ +

$- \int te^{st} dt\} =$

$- \int te^{st} dt =$

$\frac{3.2}{\ }\{t$

e^{-st}

$s \quad s \quad s^{-\infty} + \int e^{st} dt\} -s \quad 0 \quad s$

$|e = = \ _0 \quad 0 \quad 0$

$3.2^{\infty} \quad 3.2 - 1^{\infty}$

$-3.2.1$

$\infty \quad -3.2.1 \quad 3.2.1 \quad 3!$

$= \int e^{st} dt =$

$\int e^{-st}(-s) dt =$

4 $-st$

$(0 - 1) =$

So that

$s \quad 0 \quad s \quad s \quad 0 \quad s$

$L\{t^3\} = \frac{3!}{s^4}$

$0 \quad s \quad s \quad s$

Laplace Transform

The Laplace Transform of a function F(t) is denoted by L{F(t)} and is defined as the integral of F(t) e^{-st}

between the limits t=0 and t = ∞

$$L\{F(t)\}=\int_0^\infty e^{-st}F(t)dt.$$

In all cases, the constant parameter s is assumed to be positive and large enough to ensure that the product F(t) e^{-st} converges to zero as t → ∞ , what ever the function F(t).

In determining the transform of any function , you will appreciate that the limits are substituted for t, so that the result will be a function of s.

Laplace Transform of F(t) = a (constant).

That is

$$L(a)=\int_0^\infty ae^{-st}dt = a\int_0^\infty e^{-st}dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$

$$= \frac{-a}{s}\{0-1\} = \frac{a}{s}$$

Example

Find the laplace transform of the form e^{at}

that F(t)= e^{at}

where a is a constant.

$$L(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right], \quad s > a$$

$$= -\frac{1}{s-a}[0 - 1]= \frac{1}{s-a}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$s - a$

So we already have two standard transforms

$L\{a\} = \frac{a}{s}$;

$L\{e^{at}\} = \frac{1}{s-a}$;

$L\{4\} = \frac{4}{s}$;

$L\{-5\} = \frac{-5}{s}$;

$L\{e^{4t}\} = \frac{1}{s-4}$;

$L\{e^{-2t}\} = \frac{1}{s+2}$;

Laplace transform is always a function of s.

Complex Numbers Power of i

Every time a factor i^4 occurs , it can be replace by the factor 1, so that the power of i is reduced to one of the four results above.

$$i^9 = (i^4)^2 i = (1)^2 i = 1.i = i$$
$$i^{20} = (i^4)^5 = (1)^5 = 1$$
$$i^{30} = (i^4)^7 i^2 = (1)^7 (-1) = 1(-1) = -1$$
$$i^{15} = (i^4)^3 i^3 = 1(-i) = -i$$

Complex Numbers

z = 3+ 5 i , is called a complex number where 3 is real part and 5 is imaginary part of the complex number.

In general z = a+ b i , is called a complex number where a is real part and b is imaginary part of the complex number. So,

Complex Number = (Real Part)+ *i* (Imaginary Part)

Conjugate complex numbers

For a complex number a + *i* b , the complex number a - *i* b is called the conjugate of a + *i* b. Conjugate complex numbers are identical except the signs in the middle for the brackets.

- (4 + 5 i) and (4 - 5 i) are conjugate complex numbers
- (6 + 2 i) and (2 + 6 i) are not conjugate complex numbers
- (5 - 3 i) and (-5 +3 i) are not conjugate complex numbers

Remember

The product of complex number by its conjugate is always entirely real.

$(3 + 4i)(3 - 4i) = 9 + 16 = 25$

$(a + bi)(a - bi) = a^2 + b^2$

Euler Formaula

As we know that the series expansion of e^x ,cos x and sin x are given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots\ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots\ldots$$

Replace x by (it) , we get

$$e^{(it)} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \ldots\ldots\ldots\ldots.$$

$$e^{(it)} = 1 + (it) - \frac{(t)^2}{2!} - \frac{i(t)^3}{3!} + \frac{(t)^4}{4!} + \frac{i(t)^5}{5!} - \frac{(t)^6}{6!} - \frac{i(t)^7}{7!} + \ldots\ldots\ldots\ldots.$$

$$e^{(it)} = [1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \ldots\ldots\ldots\ldots.] + i[t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \quad]$$

where

$$\cos t = 1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \ldots\ldots\ldots\ldots.$$

$$\sin t = t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \ldots\ldots\ldots\ldots.$$

$$e^{it} = \cos t + i \sin t$$

R(e^{it})= cost and I(e^{it}) = sin t

The Laplace transform of F(t) = sin at

L(Sin at) = L(I(e^{iat}))

$$= I\int_0^\infty e^{iat}e^{-st}dt = I\int_0^\infty e^{-(s-ia)t}dt = I\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\}$$

$$= I\left\{\frac{1}{-(s-ia)}(0-1)\right\} = I\left\{\frac{1}{(s-ia)}\right\}$$

$$= I\left\{\frac{s+ia}{s^2+a^2}\right\} = I\left[\frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\right]$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$\overline{} s^2 + a^2$

$\therefore L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

We can use the same method to determined to determine L{cos at }.

Since cos at is the real part of e^{iat} , written as R(e^{iat})

$L\{\cos at\} = R\{\dfrac{s + ia}{s^2 + a^2}\} = R[\dfrac{s}{s^2 + a^2} + i\dfrac{a}{s^2 + a^2}]$

$L\{\cos at\} = \dfrac{s}{s^2 + a^2}$

The Transform of F(t) = t^n where n is a positive integer.

By the definition $L(t^n) = \int_0^\infty t^n e^{-st}\,dt$

integrating by parts

$L(t^n) = \int_0^\infty t^n e^{-st}\,dt = \left[t^n \frac{e^{-st}}{-s} \right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt = -\frac{1}{s}\left[t^n e^{-st} \right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$

$\therefore \left[t^n \frac{e^{-st}}{-s} \right]_0^\infty = 0 - 0 = 0$

$\therefore L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt \qquad (1)$

you will notice that $\int_0^\infty t^{n-1} e^{-st}\,dt$

is identical to $\int_0^\infty t^n e^{-st}\,dt$

except that n is replaced by (n-1)

If $\quad if\ I_n = \int_0^\infty t^n e^{-st}\,dt, then\ I_{n-1} = \int_0^\infty t^{n-1} e^{-st}\,dt$

and the result (1) becomes

$I_n = \frac{n}{s} I_{n-1}$

This is reduction formula and , if we now replace n by (n-1) we get

$I_{n-1} = \frac{n-1}{s} I_{n-2}$

If we replace n by (n-1) again in this last result , we have

$I_{n-2} = \frac{n-2}{s} I_{n-3}$

∞

$n \quad n\ n-1$

$n\ n-1$

$n-2$

$n\ n-1\ n-2$

$n-3$

$So\ I_n = \int_0^\infty t^n e^{-st}\,dt =$

$I_{n-1} =.$

$I_{n-2} = \quad . \quad .$

$I_{n-3} =$

$. \quad . \quad .$

I_{n-4}

So finally, we have

$s \quad s \quad s \quad s \quad s \quad s \quad s \quad s \quad s \quad s$

$I_n = \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} . \frac{n-4}{s} \quad \frac{2}{s}\ \frac{1}{s}$

But

$\ldots\ldots\ldots\ldots\ldots . I \quad s \quad s \quad s \quad s \quad s \quad s \; s \; 0$

$I_0 = L\{t^0\} = L\{1\} = \frac{1}{s}$

$I_n = \frac{n}{s}\;\frac{n-1}{s}\;\frac{n-2}{s}\;\frac{n-3}{s}\;\frac{n-4}{s} \ldots \frac{}{s\; s} I_0$

$= \frac{n!}{s^{n+1}}$

$L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{t\} = \frac{1}{s^2}$; $L\{t^2\} = \frac{2!}{s^3}$; $L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$

Laplace Transform of F(t) = sinh at and F(t) = cosh at.

Starting from the exponential definitions of sinh at and cosh at

i.e. $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$ *and* $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$

We proceed as follow

a) $F(t) = \sinh at$

$\text{Sinh at} = \int_0^{\infty} \sinh at\, e^{-st} dt = \frac{1}{2}\int_0^{\infty} (e^{at} - e^{-at}) e^{-st} dt$

$= \frac{1}{2}\int_0^{\infty} (e^{at}e^{-st} - e^{-at}e^{-st}) dt = \frac{1}{2}\int_0^{\infty} (e^{-(s-a)t} - e^{-(s+a)t}) dt$

$\frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} - \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}\right\} = \frac{1}{2}\left\{\frac{1}{-(s-a)}(0-1) - \frac{1}{-(s+a)}(0-1)\right\} = \frac{1}{2}\left\{\frac{1}{(s-a)} - \frac{1}{(s+a)}\right\}$

2 $= 2\begin{cases} \dfrac{1}{2}\left[\dfrac{s+a-s+a}{(s-a)(s+a)}\right] = \dfrac{1}{2}\left[\dfrac{2a}{s}\right] = \left[\dfrac{a}{}\right] \end{cases}$

b) $F(t) = \cosh at$

$a \rfloor \quad \lfloor \; s$

$a \rfloor$

$$L(\cosh at) = L\left(\frac{1}{2}\left(e^{at} + e^{-at}\right)\right) = \frac{1}{2}\int_0^\infty (e^{at} + e^{at})e^{-st}dt = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)t}dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right\} = \frac{1}{2}\left\{\frac{1}{-(s-a)}(0-1) + \frac{1}{-(s+a)}(0-1)\right\} = \frac{1}{2}\left\{\frac{1}{(s-a)} + \frac{1}{(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+a+s-a}{(s-a)(s+a)}\right\} = \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\} = \left\{\frac{s}{s^2-a^2}\right\}$$

$$\therefore L(\cosh at) = \left\{\frac{s}{s^2-a^2}\right\}$$

Several Standard Results

$L\{a\} = \frac{a}{s}$;

$L\{e^{at}\} = \frac{1}{s-a}$;

$L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$;

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$;

$L\{\cosh at\} =$

s

$s^2 - a^2$

We can , of course, combine these transforms by adding or subtracting as necessary , but they must not be multiplied together to form the transform of a product.

Example

a) $L\{2\sin 3t + \cos 3t\} = 2L(\sin 3t) + L(\cos 3t) = 2.$

3
$+$
$s^2 + 9$

s
$=$
$s^2 + 9$

$\dfrac{s+6}{s^2+9}$

b) $L\{4e^{2t} + 3\cosh 4t\} = 4L(e^{2t}) + L(3\cosh 4t) = 4.$
s

1
$+3.$
-2

s
$=$
$s^2 - 16 \quad s$

$4 \qquad 3s$
$+$
$-2 \qquad s^2 - 16$

$$= \frac{7s^2 - 6s - 64}{(s-2)(s^2-16)}$$